Lecture 30

Examples of symmetric spaces of noncpt type.
These will be irreducible so G = semisimp K = max cpt g = Killing.

1) Hyperbolic spaces $\mathbb{H}^n = SO(n,1)/SO(n)$ ($\mathbb{H}^2$ = upper half plane)
$= \mathbb{B}^n$ w/ met $\dfrac{4\,ds^2_{euc}}{(1-r^2)^2}$

2) Complex hyp spaces $\mathbb{C}\mathbb{H}^n = SU(n,1)/SU(n)$ ($\mathbb{C}\mathbb{H}^1$ = unit disk)
$\cong \mathbb{B}^{2n}$ in $\mathbb{C}^n$ w/ metric.

3) $\mathrm{Inner}_0(\mathbb{R}^n) = \{$positive definite inner prod s.t. $[-1,1]^n$ has volume $2^n\}$
Such an inner product always has the form $\langle x,y\rangle_A = (Ax)^T(Ay)$
for $A \in SL_n\mathbb{R}$. A is not unique because if $\quad = x^T A^T A x$
$M^TM = \mathrm{Id}$, then A and MA have the same assoc $\langle,\rangle$

So $\langle,\rangle_A \longmapsto SO(n)\cdot A$ gives $\mathrm{Inner}_0(\mathbb{R}^n) \cong \cancel{SO(n) \backslash SL_n\mathbb{R}}$

The action: $(T\cdot\langle,\rangle_A)(x,y) = \langle Tx,Ty\rangle_A$ is a <u>right</u> action
$(S\cdot(T\cdot\langle,\rangle_A))(x,y) = (T\cdot\langle,\rangle_A)(Sx,Sy) = \langle TSx, TSy\rangle_A$
$= ((TS)\cdot\langle,\rangle_A)(x,y)$.

Fix. Define $\langle,\rangle_A$ using A^{-1}.
$T\cdot\langle,\rangle_A = \langle T^{-1}x, T^{-1}y\rangle$

$\mathrm{Inner}_0 \cong SL_n\mathbb{R}/SO(n)$
w/ left mul by $SL_n\mathbb{R}$

Cartan: $\mathfrak{sl}_n\mathbb{R} = \mathfrak{k} \oplus \mathfrak{p}$ $\quad \mathfrak{k} = \{X + X^T = 0\}$ $\quad \mathfrak{p} = \{X - X^T = 0\}$
$X = X^T$.

$B(X,Y) = \mathrm{tr}(XY)$ up to scale. Check: neg def on $\mathfrak{k}$ $\mathfrak{os}$

$tr(XY) =$ sum of (row i X)$\cdot$(col i of Y) if $X = -X^T$

$tr(XX) =$ sum $\underline{\qquad\qquad}$ $=$ $-$(row i)$\cdot$(row i) ≤ 0.

Metric interp: Given an inner product $\langle , \rangle = p$.

$v =$ tgt vec of $\langle (I - \varepsilon X)\cdot, (I-\varepsilon X)\cdot \rangle$ where $X = X^T$

$w =$ $\underline{\qquad\qquad}$ $\langle (I-\varepsilon Y)\cdot, (I-\varepsilon Y)\cdot \rangle$ $Y = Y^T$.

Then $g_p(v,w) = tr(XY)$.

Exercise. $\gamma(t) =$ inner product where e_1, e_2, e_3 orthog.

$\|e_1\| = a(t)$ $\|e_2\| = b(t)$ $\|e_3\| = \frac{1}{a(t)b(t)}$.

Find length $\gamma(t)$ $t \in [0,1]$.

Geom interp. $\mathrm{Inner}_0(\mathbb{R}^n) \cong \mathrm{Ell}_0(\mathbb{R}^n)$ ellipsoids of unit vol.

B is unit ball $\leftarrow\!\shortmid$ B

$\langle , \rangle \longmapsto \{ v \mid \langle v, v \rangle \leq 1 \}$.

Tangent vec at $\sum x_i^2 = 1$: $\sum x_i^2 + \varepsilon(\text{homog poly deg } 2) = 1$.

Length of this vector is the Euclidean norm of the vector of coefs of that poly.

4) $\mathrm{Herm}(\mathbb{C}^n) = \{$hermitian, pos def inner on $\mathbb{C}^n$ s.t. polydisk Δ^n has volume $\pi^n \}$

$\cong SL_n\mathbb{C}/SU(n)$ by $\langle z, w \rangle = (\overline{A^{-1}z})^t (A^{-1}w) \longleftrightarrow A\, SU(n)$

$\mathfrak{sl}_n\mathbb{C} = \mathfrak{k} \oplus \mathfrak{p}$ $\mathfrak{p} = i\mathfrak{k}$ real part of the complex killing form.

Low dim accident: $PSL_2\mathbb{C} \cong SO_0(3,1) \rightsquigarrow \underbrace{\mathrm{Herm}(\mathbb{C}^2)}_{\text{Hermitian matrix model.}}$ iso to $\underbrace{\mathbb{H}^3}_{\text{Minkowski model}}$.

Curvature

$R_p(v,w)$ is a linear operator on T_pM (Riem curv tensor)

On a symm space, if we identify tangent spaces with $\mathfrak{p}$ up to $\mathfrak{k}_\gamma$,

$$R(x,y) = \mathrm{ad}_{\underbrace{[x,y]}_{\mathfrak{k}_\gamma}} : \mathfrak{p} \to \mathfrak{p}.$$

i.e. $g(R(x,y)z, w) = B([[x,y],z], w)$

Has positive curvature : $g(R(v,w)(v),w) > 0 \quad \forall v,w$

etc.

For G/K noncpt type: $B([[x,y],x], y)$

$[[x,y],x] + \cancel{[[x,x],y]} + [[y,x],x] = 0$

$[[x,y],x] = -[[y,x],x] = [x,[y,x]] = -[x,[x,y]]$

$B([[x,y],x],y) = -B([x,[x,y]],y) = -B(\mathrm{ad}_x[x,y], y)$

$= B([x,y], \mathrm{ad}_x(y)) = B([x,y],[x,y])$

Now $[x,y] \in \mathfrak{k}_\gamma$ so this is ≤ 0.

Cor Symm spaces of noncpt type have nonpos curvature.

Cor. Symm sp of noncpt type diffeo to $\mathbb{R}^n$.

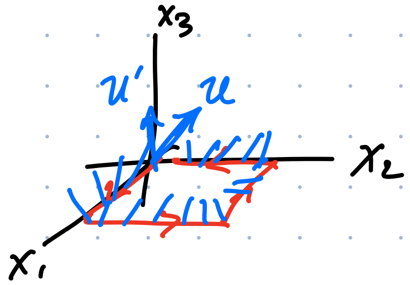
Geometric interp of $R_p(v,w)$.

Key concept is parallelism: $g \rightsquigarrow$ For smooth path γ from p to q, get $P_\gamma : T_pM \to T_qM$ isometry. Depends on γ.

In fact, path-indep $\iff$ $R_p(v,w)=0$ $\forall$ p, v, w.

Now in local coord where $p=0$, $v=\left(\frac{\partial}{\partial x_1}\right)_0$ $w=\left(\frac{\partial}{\partial x_2}\right)_0$

let us form a rectangle of side length ε parallel to x_1 and x_2 axes.



let P_ε be the parallel transport op.

$$R_p(v,w) = \lim_{\varepsilon \to 0} \frac{u - P_\varepsilon(u)}{\varepsilon^2}.$$

Flats and Rank.

A subspace $s \subset \mathfrak{g}$ is called a Lie triple system if $X, Y, Z \subset s \Rightarrow [X,[Y,Z]] \subset s$.

Prop. Let $s \subset \mathfrak{p}$ be triple system. Then $S = \underbrace{\mathrm{Exp}}_{\text{Riemannian exp}}(s)$ is totally geodesic (geod in it are geod in G/K).